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Generalized Q-functions

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Abstract
The modulus squared of a class of wavefunctions defined on phase space is used to define a generalized family of Q or Husimi functions. A parameter λ specifies orderings in a mapping from the operator $|\psi\rangle\langle\sigma|$ to the corresponding phase space wavefunction, where σ is a given fiducial vector. The choice $\lambda = 0$ specifies the Weyl mapping and the Q-function so obtained is the usual one when $|\sigma\rangle$ is the vacuum state. More generally, any choice of λ in the range $(-1, 1)$ corresponds to orderings varying between standard and anti-standard. For all such orderings the generalized Q-functions are non-negative by construction. They are shown to be proportional to the expectation of the system state $\hat{\rho}$ with respect to a generalized displaced squeezed state which depends on λ and position (p, q) in phase space. Thus, when a system has been prepared in the state $\hat{\rho}$, a generalized Q-function is proportional to the probability of finding it in the generalized squeezed state. Any such Q-function can also be written as the smoothing of the Wigner function for the system state $\hat{\rho}$ by convolution with the Wigner function for the generalized squeezed state.

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1. Introduction

The Weyl transform [1, 2] associates operators with functions on phase space. In particular, the Wigner function $\rho(p, q)$ [3] is the Weyl transform of the density matrix divided by $h = 2\pi\hbar$. Although $\rho(p, q)$ does have many features of a classical distribution it can take on negative values, with bounds [5] given by $2/h \geqslant \rho(p, q) \geqslant -2/h$. Indeed Hudson [4] showed that the only pure states $\psi(x)$ for which the Wigner function is non-negative are Gaussian in x. This carries over to any number of dimensions [6], and also, for odd dimensions at least, to the formulation of discrete Wigner functions [7].

The Weyl correspondence between operators and functions on phase space—of which the Wigner function is an example—is a special case of the class of the correspondences given by Cohen [8]. In particular, if any Wigner function is convoluted, or smeared, by integration with respect to the Wigner function of the vacuum state, itself a gaussian function on phase space, then the smoothed function, called the Q-function (or Husimi function), is non-negative and corresponds to an ordering in Cohen's class different from that of Wigner and Weyl [9]. More generally, if any Wigner function is convoluted with respect to a Gaussian function which is itself the Wigner function of a pure coherent state, then the result is non-negative [6, 9–13].

The Wigner function is bilinear with respect to wavefunctions. For instance if the Weyl transform of the pure state $|\psi\rangle\langle\psi|$ is written $(|\psi\rangle\langle\psi|)_{(p,q)}$, then the corresponding Wigner function [3, 5] is

$$\begin{aligned}\rho(p,q) &= \frac{1}{h}(|\psi\rangle\langle\psi|)_{(p,q)} \\ &= \frac{1}{h}\int_{-\infty}^{\infty}\mathrm{d}x\,\exp\left(\frac{\mathrm{i}}{\hbar}px\right)\psi\left(q-\frac{x}{2}\right)\psi^{*}\left(q+\frac{x}{2}\right),\end{aligned} \tag{1}$$

so the smeared Wigner functions are also bilinear with respect to the wavefunctions.

It is also possible in a sense to smear the states themselves, for instance by projecting them onto a class of generalized displaced coherent states, defined [14] by

$$|p,q;\sigma\rangle \equiv \hat{D}[p,q]|\sigma\rangle, \tag{2}$$

where $|\sigma\rangle$ is any reference 'fiducial' state, and

$$\hat{D}[p,q] = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p\hat{q}-q\hat{p})} \tag{3}$$

is Weyl's displacement operator. Then, corresponding to any wavefunction $|\psi\rangle$ one can define a 'smoothed' wavefunction on phase space by projecting it onto the coherent state:

$$\widetilde{\psi}_{\sigma}(p,q) \equiv \langle\sigma|\hat{D}^{\dagger}[p,q]|\psi\rangle. \tag{4}$$

These functions and their time dependence when ψ is driven by the Hamiltonian $\hat{p}^2/2m + V(q)$ have been studied for some choices of $|\sigma\rangle$ by Torres-Vega *et al*, Harriman, and others [15–17].

In this paper I generalize $\widetilde{\psi}_{\sigma}(p,q)$ to a phase space wavefunction $\widetilde{\psi}_{\sigma}^{(\lambda)}(p,q)$ by relating it to a class of orderings labelled by a parameter $\lambda \in (-1,+1)$, where $\widetilde{\psi}_{\sigma}^{(0)}(p,q) = \widetilde{\psi}_{\sigma}(p,q)$, equation (4). A given value of λ specifies an association between functions on phase space and operators, $A(p,q) \overset{(\lambda)}{\longleftrightarrow} \hat{A}$, where $\lambda = -1$ gives the standard ordering (e.g. $p^n q^m \longleftrightarrow \hat{q}^m\hat{p}^n$), $\lambda = +1$ gives the anti-standard rule (e.g. $p^n q^m \longleftrightarrow \hat{p}^n\hat{q}^m$), and $\lambda = 0$ gives the symmetric or Weyl association, of which (1) is an example with $\rho(p,q) \longleftrightarrow \hat{\rho}/h$. The time dependence, effectively, of $\widetilde{\psi}_{\sigma}^{(\lambda)}(p,q)$ has been studied in [18].

$\widetilde{\psi}_{\sigma}^{(\lambda)}(p,q)$ relates to the λ-orderings of the operator $|\psi\rangle\langle\sigma|$, which is linear in the states $|\psi\rangle$ (the reference or fiducial state is held fixed), but the density matrix $\hat{\rho} = |\psi\rangle\langle\psi|$ is bilinear, so a chosen ordering for $|\psi\rangle\langle\sigma|$ will not be expected to apply to the density matrix, indeed it may not even be of the λ-class. The generalized Q-function for a pure state $|\lambda\rangle$, defined as $\left|\widetilde{\psi}_{\sigma}^{(\lambda)}(p,q)\right|^2\big/h$, is normalized with respect to the integral $\int \mathrm{d}p\mathrm{d}q$ over all of phase space. The main results of this paper are that the generalized Q-function corresponding to any state $\hat{\rho}$ is, first, non-negative, second, proportional to the expectation of ρ with respect to a certain generalized displaced squeezed state which depends upon σ, λ and (p,q) and, third, proportional to the convolution of the Wigner functions for ρ with the Wigner function for that squeezed state.

The field of quantum mechanics in phase space is a large one, perhaps starting with the analysis of Weyl [1, 2] and of Wigner [3]. In the context of this paper Bopp [19] in 1956 considered classical-like implications of that Q-function corresponding to the Weyl ordering

($\lambda = 0$) and with fiducial state chosen (as is usually the case) to be the vacuum state $|0\rangle \equiv |h_0\rangle$, namely $\langle h_0|\hat{D}[p,q]^\dagger\hat{\rho}(t)\hat{D}[p,q]|h_0\rangle$. That this can be related to the modulus squared of a wavefunction, here $\widetilde{\psi}^{(0)}_{h_0}(p,q)$ was pointed out by Mizrahi [20] who also studied some of its properties. On a different tack, Cahill and Glauber [21, 22] have studied at length a family of orderings (the s-family) $\mathbf{A} \overset{(s)}{\longleftrightarrow} A(p,q)$, centred around the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, where (in my notation) $\hat{a} = \frac{1}{\sqrt{2}}\big(\alpha\hat{q} + \mathrm{i}\frac{\hat{p}}{\alpha\hbar}\big)$—where α is a real parameter—so that $[\hat{a},\hat{a}^\dagger] = 1$. Defining the complex numbers $\mathcal{A} = \frac{1}{\sqrt{2}}\big(\alpha q + \mathrm{i}\frac{p}{\alpha\hbar}\big)$, when $s = -1$ their mapping corresponds to the association (antinormal ordering) $\hat{a}^m\hat{a}^{\dagger n} \longleftrightarrow \mathcal{A}^m\mathcal{A}^{*n}$, when $s = 1$ the association is $\hat{a}^{\dagger m}\hat{a}^n \longleftrightarrow \mathcal{A}^{*m}\mathcal{A}^n$ (normal ordering), and when $s = 0$ the ordering is that of Weyl. Thus the λ and s mappings complement each other, and overlap at $\lambda = 0 = s$. Among their many interesting results Cahill and Glauber define what is effectively a phase space wavefunction corresponding to $|\psi\rangle\langle h_0|$ for their s-ordering, but they do not relate its modulus squared to any s-ordered Q-function. They do, however, express the usual Q-function, $\langle h_0|\hat{D}[p,q]^\dagger\hat{\rho}(t)\hat{D}[p,q]|h_0\rangle$, as a smoothed Wigner function.

In this note I start with the modulus squared of wavefunctions on phase space, of which equation (4) is a special case, and show that it can correspond to smeared Wigner functions, where the smearing functions themselves are Wigner functions of generalized displaced squeezed states. Section 2 discusses wavefunctions on phase space and generalizes them to the λ-class of orderings. Section 3 develops expressions for the Q-functions based on these wavefunctions. Section 4 discusses some properties of these Q-functions.

2. Wavefunctions on phase space

It is often convenient to work with the Fourier transform of $\widetilde{\psi}_\sigma(p,q)$, defined by

$$\begin{aligned}\psi_\sigma(p,q) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}p'\,\mathrm{d}q'}{h}\exp\left[\frac{\mathrm{i}}{\hbar}(p'q - q'p)\right]\widetilde{\psi}_\sigma(p',q')\\ &= \mathrm{Tr}(|\psi\rangle\langle\sigma|\hat{\Delta}(p,q)),\end{aligned}\tag{5}$$

where [5]

$$\begin{aligned}\hat{\Delta}(p,q) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}p'\,\mathrm{d}q'}{h}\exp\left[-\frac{\mathrm{i}}{\hbar}(p'q - q'p)\right]\hat{D}[p',q']\\ &= \int_{-\infty}^{\infty}\mathrm{d}x\,\exp\left(\frac{\mathrm{i}}{\hbar}px\right)\left|q + \frac{x}{2}\right\rangle\left\langle q - \frac{x}{2}\right|.\end{aligned}\tag{6}$$

The wavefunctions $\psi_\sigma(p,q)$ were defined in [18] where many of their properties are discussed. In particular, they are the Weyl transform of the operators $|\psi\rangle\langle\sigma|$. Indeed, the Weyl transform, which I shall write $(\hat{A})_{(p,q)}$ or $A_{(p,q)}$, and its associated operator $\hat{A}$ are related [5] by

$$\hat{A} = \int_{-\infty}^{\infty}\frac{\mathrm{d}p\,\mathrm{d}q}{h}A_{(p,q)}\hat{\Delta}(p,q),\tag{7}$$

which, by virtue of the relation

$$\mathrm{Tr}(\hat{\Delta}(p,q)\hat{\Delta}(p',q')) = h\delta(p-p')\delta(q-q'),\tag{8}$$

can be inverted to give

$$A_{(p,q)} = \mathrm{Tr}(\hat{A}\hat{\Delta}(p,q)).\tag{9}$$

So $\psi_\sigma(p,q)$ is the Weyl transform $(|\psi\rangle\langle\sigma|)_{(p,q)}$, and $\widetilde{\psi}_\sigma(p,q)$ is its Fourier transform.

Another property of the Weyl transform which we need [5] is

$$\mathrm{Tr}(\hat{A}\hat{B}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p\,\mathrm{d}q}{h} A_{(p,q)} B_{(p,q)}. \tag{10}$$

Note from (6) that $\mathrm{Tr}(\hat{\Delta}(p,q)) = 1$ so, from (9), $(\hat{1})_{(p,q)} = 1$, and (letting $\hat{B} = \hat{1}$ in (10))

$$\mathrm{Tr}\,\hat{A} = \int_{-\infty}^{\infty} \frac{\mathrm{d}p\,\mathrm{d}q}{h} A_{(p,q)}. \tag{11}$$

The essential characteristic of the Weyl correspondence follows from equations (3) and (9) together with the first of (6). It is

$$(\mathrm{e}^{\mathrm{i}(\xi\hat{q}+\eta\hat{p})})_{(p,q)} = \mathrm{e}^{\mathrm{i}(\xi q+\eta p)}. \tag{12}$$

Other orderings defined by Cohen [8] can be specified by the generalization of (12) to the form

$$(\mathrm{e}^{\mathrm{i}(\xi\hat{q}+\eta\hat{p})})^{f}_{(p,q)} = \frac{1}{f(\xi,\eta)}\,\mathrm{e}^{\mathrm{i}(\xi q+\eta p)} = f^{-1}(-\mathrm{i}\partial_q, -\mathrm{i}\partial_p)\,\mathrm{e}^{\mathrm{i}(\xi q+\eta p)}, \tag{13}$$

where f^{-1} means $1/f$ and the choice $f = 1$ gives the Wigner–Weyl ordering. Note that when $f(0,\eta) = 1 = f(\xi,0)$ then the Weyl transform of a function of $\hat{q}$ (or $\hat{p}$) only is the same function of q (or p) only. If we particularize to the class of orderings defined by the function

$$f(\xi,\eta;\lambda) = \mathrm{e}^{\mathrm{i}\frac{\hbar}{2}\lambda\xi\eta}, \tag{14}$$

where λ is a real parameter lying in the interval $[-1, +1]$, then

$$(\mathrm{e}^{\mathrm{i}(\xi\hat{q}+\eta\hat{p})})^{(\lambda)}_{(p,q)} = \mathrm{e}^{-\mathrm{i}\frac{\hbar}{2}\lambda\xi\eta}\,\mathrm{e}^{\mathrm{i}(\xi q+\eta p)}. \tag{15}$$

Use of the Baker–Campbell–Hausdorff theorem leads to the equivalent expressions

$$(\mathrm{e}^{\mathrm{i}\xi\hat{q}}\,\mathrm{e}^{\mathrm{i}\eta\hat{p}})^{(\lambda)}_{(p,q)} = \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}(\lambda+1)\xi\eta}\,\mathrm{e}^{\mathrm{i}(\xi q+\eta p)}$$

and

$$(\mathrm{e}^{\mathrm{i}\eta\hat{p}}\,\mathrm{e}^{\mathrm{i}\xi\hat{q}})^{(\lambda)}_{(p,q)} = \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}(\lambda-1)\xi\eta}\,\mathrm{e}^{\mathrm{i}(\xi q+\eta p)}.$$

The choice $\lambda = -1$ in the first of these gives the 'standard' or 'p' association ($\hat{p}$ first, then $\hat{q}$),

$$(\mathrm{e}^{\mathrm{i}\xi\hat{q}}\,\mathrm{e}^{\mathrm{i}\eta\hat{p}})^{(-1)}_{(p,q)} = \mathrm{e}^{\mathrm{i}(\xi q+\eta p)}$$

and the choice $\lambda = 1$ in the second gives the ànti-standard association ($\hat{q}$ first, then $\hat{p}$),

$$(\mathrm{e}^{\mathrm{i}\eta\hat{p}}\,\mathrm{e}^{\mathrm{i}\xi\hat{q}})^{(+1)}_{(p,q)} = \mathrm{e}^{\mathrm{i}(\xi q+\eta p)},$$

while the Wigner–Weyl ordering, $\lambda = 0$, puts $\hat{p}$ and $\hat{q}$ on equal footing, equation (12).

The generalization of $\psi_\sigma(p,q)$ to the family of orderings defined by equations (14) and (15) is given [18] by

$$\psi^{(\lambda)}_\sigma(p,q) = \mathrm{Tr}(|\psi\rangle\langle\sigma|\hat{\Delta}^{(\lambda)}(p,q)) = \langle\sigma|\hat{\Delta}^{(\lambda)}(p,q)|\psi\rangle, \tag{16}$$

where

$$\hat{\Delta}^{(\lambda)}(p,q) = \mathrm{e}^{\mathrm{i}\frac{\hbar}{2}\lambda\partial_p\partial_q}\,\hat{\Delta}(p,q). \tag{17}$$

Equations (16) and (17) generalize the phase space wavefunction $\psi_\sigma(p,q)$, the Weyl transform of $|\psi\rangle\langle\sigma|$, to the class of orderings defined by (14).

3. Q-functions

The functions $\psi_\sigma(p,q)$ are normalized—this follows from the second of equations (5) and (10)—and so too are the $\widetilde{\psi}_\sigma(p,q)$ by dint of the Fourier relation, equation (5). Further, by taking matrix elements of the quantities in equations (3) and (7) one finds [23] that

$$\hat{\Delta}(p,q) = 2\hat{D}[2p,2q]\hat{\Pi} \qquad \text{or} \qquad \hat{D}[p,q] = \tfrac{1}{2}\hat{\Delta}(p/2,q/2)\hat{\Pi}, \tag{18}$$

where $\hat{\Pi}$ is the parity operator, i.e.

$$\hat{\Pi} = \int_{-\infty}^{\infty} \mathrm{d}x|x\rangle\langle -x|. \tag{19}$$

From these equations we can define a generalized displacement operator as

$$\hat{D}^{(\lambda)}[p,q] = \tfrac{1}{2}\hat{\Delta}^{(\lambda)}(p/2,q/2)\hat{\Pi} \tag{20}$$

with corresponding generalized 'coherent state' $\hat{D}^{(\lambda)}[p,q]|\sigma\rangle$ and phase space wavefunction (partner and equivalent to $\psi_\sigma^{(\lambda)}(p,q)$) given by

$$\widetilde{\psi}_\sigma^{(\lambda)}(p,q) = \langle\sigma|\hat{D}^{(\lambda)\dagger}[p,q]|\psi\rangle. \tag{21}$$

Consider the product

$$\begin{aligned}\left(\mu_\sigma^{(\lambda)}(p,q)\right)^*\psi_\sigma^{(\lambda)}(p,q) &= \int \mathrm{d}\tau' \int \mathrm{d}\tau''\, \mathrm{e}^{\mathrm{i}\frac{\lambda}{2\hbar}p'q'}\mathrm{e}^{-\mathrm{i}\frac{\lambda}{2\hbar}p''q''} \\ &\quad\times \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p'q-q'p)}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(p''q-q''p)}\widetilde{\psi}_\sigma(p',q')(\widetilde{\mu}_\sigma(p'',q''))^*,\end{aligned} \tag{22}$$

where I have used (5), (16) and (17) and $\int \mathrm{d}\tau'$ stands for $\int_{-\infty}^{\infty}\mathrm{d}p'\mathrm{d}q'/h$, etc. By equations (4), (9) and (10) we can write

$$\widetilde{\psi}_\sigma(p',q')(\widetilde{\mu}_\sigma(p'',q''))^* = \int \mathrm{d}\tau (|\psi\rangle\langle\mu|)_{(p,q)}(\hat{D}(p'',q'')(|\sigma\rangle\langle\sigma|\hat{D}^\dagger(p',q'))_{(p,q)}, \tag{23}$$

which is an integral over the product of two Weyl transformed operators. In particular, by definition (9) the second term is

$$(\hat{D}(p'',q'')(|\sigma\rangle\langle\sigma|\hat{D}^\dagger(p',q'))_{(p,q)} = \langle\sigma|\hat{D}^\dagger(p',q')\hat{\Delta}(p,q)\hat{D}(p'',q'')|\sigma\rangle. \tag{24}$$

To simplify this quantity one can express $\hat{\Delta}$ here in terms of $\hat{D}$ (equation (6)) and then simplify the resulting triple product of displacement operators by means of these useful algebraic properties [14]:

$$\begin{aligned}&\hat{D}^\dagger[p,q] = \hat{D}[-p,-q],\\ &\hat{D}^\dagger[p,q](\hat{p},\hat{q})\hat{D}[p,q] = (\hat{p}+p,\hat{q}+q),\\ &\hat{D}[p_2,q_2]\hat{D}[p_1,q_1] = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}(q_1p_2-q_2p_1)}\hat{D}[p_1+p_2,q_1+q_2].\end{aligned} \tag{25}$$

Utilizing the action of the unitary operator $\hat{D}$ on $\hat{\Delta}$ itself can also help. For instance, using the second of (25) with the first of equations (6) one finds

$$\hat{D}^\dagger[p',q']\hat{\Delta}(p,q)\hat{D}[p',q'] = \hat{\Delta}(p-p',q-q'). \tag{26}$$

The upshot is that by direct calculation equations (22) to (26) can be combined and simplified to give

$$\left(\mu_\sigma^{(\lambda)}(p,q)\right)^*\psi_\sigma^{(\lambda)}(p,q) = \left(\frac{4}{1-\lambda^2}\right)\int \mathrm{d}\tau'(|\psi\rangle\langle\mu|)_{(p',q')}(|\sigma\rangle\langle\sigma|)_{\left(\frac{2p-(1+\lambda)p'}{(1-\lambda)},\frac{2q-(1-\lambda)q'}{(1+\lambda)}\right)}. \tag{27}$$

It is easy to see from this result that

$$\int \mathrm{d}\tau \left(\mu_\sigma^{(\lambda)}(p,q)\right)^* \psi_\sigma^{(\lambda)}(p,q) = \langle\mu|\psi\rangle, \tag{28}$$

as it must [18].

From (27) we can find an analogous expression for the pair $\left(\widetilde{\psi}_\sigma^{(\lambda)}, \widetilde{\mu}_\sigma^{(\lambda)}\right)$. By equations (21), (10) and (20) it is

$$\begin{aligned}
\left(\widetilde{\mu}_\sigma^{(\lambda)}(p,q)\right)^* \widetilde{\psi}_\sigma^{(\lambda)}(p,q) &= \mathrm{Tr}(|\psi\rangle\langle\mu|\hat{D}^{(\lambda)}[p,q]|\sigma\rangle\langle\sigma|\hat{D}^{(\lambda)\dagger}[p,q]) \\
&= \frac{1}{4}\int \mathrm{d}\tau'(|\psi\rangle\langle\mu|)_{(p',q')} \\
&\quad \times \langle\sigma|\hat{\Pi}\hat{\Delta}^{(\lambda)\dagger}(p/2,q/2)\hat{\Delta}(p',q')\hat{\Delta}^{(\lambda)}(p/2,q/2)\hat{\Pi}|\sigma\rangle \\
&= \frac{1}{4}\int \mathrm{d}\tau'(|\psi\rangle\langle\mu|)_{(p',q')}\langle\sigma|\hat{\Delta}^{(\lambda)\dagger} \\
&\quad \times (-p/2,-q/2)\hat{\Delta}(-p',-q')\hat{\Delta}^{(\lambda)}(-p/2,-q/2)|\sigma\rangle,
\end{aligned} \tag{29}$$

where I have recognized (using $\hat{\Pi}$ with the first of equations (6)) that

$$\hat{\Pi}\hat{\Delta}^{(\lambda)}(p,q)\hat{\Pi} = \hat{\Delta}^{(\lambda)}(-p,-q).$$

Similarly (use an analysis based on (16))

$$\begin{aligned}
\left(\mu_\sigma^{(\lambda)}(p,q)\right)^* \psi_\sigma^{(\lambda)}(p,q) &= \int \mathrm{d}\tau'(|\psi\rangle\langle\mu|)_{(p',q')} \\
&\quad \times\langle\sigma|\hat{\Delta}^{(\lambda)}(p,q)\hat{\Delta}(p',q')\hat{\Delta}^{(\lambda)\dagger}(p,q)|\sigma\rangle.
\end{aligned} \tag{30}$$

Since $\hat{\Delta}^{(\lambda)\dagger}(p,q) = \hat{\Delta}^{(-\lambda)}(p,q)$ it follows from (29) and (30) that multiplying by 1/4 and making the substitutions $(p,q,p',q',\lambda) \to (-p/2,-q/2,-p',-q',-\lambda)$ in (27) gives

$$\left(\widetilde{\mu}_\sigma^{(\lambda)}(p,q)\right)^* \widetilde{\psi}_\sigma^{(\lambda)}(p,q) = \left(\frac{1}{1-\lambda^2}\right)\int \mathrm{d}\tau'(|\psi\rangle\langle\mu|)_{(p',q')}(|\sigma\rangle\langle\sigma|)_{\left(\frac{(1-\lambda)p'-p}{(1+\lambda)},\frac{(1+\lambda)q'-q}{(1-\lambda)}\right)}. \tag{31}$$

This also obeys an equation like (28).

The second term in the integrand here is the Weyl transform of the pure state $|\sigma\rangle\langle\sigma|$, namely, from (9),

$$(|\sigma\rangle\langle\sigma|)_{\left(\frac{(1-\lambda)p'-p}{(1+\lambda)},\frac{(1+\lambda)q'-q}{(1-\lambda)}\right)} = \langle\sigma|\hat{\Delta}\left(\frac{(1-\lambda)p'-p}{(1+\lambda)},\frac{(1+\lambda)q'-q}{(1-\lambda)}\right)|\sigma\rangle.$$

This can be simplified using the displacement operator $\hat{D}$ and the unitary dilation, or squeeze, operator ([24, 25])

$$\hat{S}(\xi) = \mathrm{e}^{\mathrm{i}\frac{\xi}{2\hbar}(\hat{p}\hat{q}+\hat{q}\hat{p})}, \tag{32}$$

which has the properties

$$\hat{S}^\dagger(\xi) = \hat{S}(-\xi) \qquad \text{and} \qquad \hat{S}^\dagger(\xi)(\hat{p},\hat{q})\hat{S}(\xi) = (\mathrm{e}^{\xi}\hat{p},\mathrm{e}^{-\xi}\hat{q}), \tag{33}$$

so that (using this with (3) and (6))

$$\hat{S}^\dagger(\xi)\hat{\Delta}(p,q)\hat{S}(\xi) = \hat{\Delta}(\mathrm{e}^{-\xi}p,\mathrm{e}^{\xi}q). \tag{34}$$

Then

$$\begin{aligned}
(|\sigma\rangle\langle\sigma|)_{\left(\frac{(1-\lambda)p'-p}{(1+\lambda)},\frac{(1+\lambda)q'-q}{(1-\lambda)}\right)} &= \langle p,q,\lambda;\sigma|\Delta(p',q')|p,q,\lambda;\sigma\rangle \\
&= (|p,q,\lambda;\sigma\rangle\langle p,q,\lambda;\sigma|)_{(p',q')},
\end{aligned} \tag{35}$$

where

$$|p,q,\lambda;\sigma\rangle = \hat{D}\left[\frac{p}{1-\lambda},\frac{q}{1+\lambda}\right]\hat{S}\left(\ln\frac{1+\lambda}{1-\lambda}\right)|\sigma\rangle \tag{36}$$

is a displaced squeezed state [14, 24, 25] generalized to an arbitrary fiducial state $|\sigma\rangle$. And so

$$\begin{aligned}\left(\widetilde{\mu}^{(\lambda)}_\sigma(p,q)\right)^*\widetilde{\psi}^{(\lambda)}_\sigma(p,q) &= \left(\frac{1}{1-\lambda^2}\right)\int \mathrm{d}\tau'(|\psi\rangle\langle\mu|)_{(p',q')}(|p,q,\lambda;\sigma\rangle\langle p,q,\lambda;\sigma|)_{(p',q')}\\ &= \left(\frac{1}{1-\lambda^2}\right)\langle p,q,\lambda;\sigma|\psi\rangle\langle\mu|p,q,\lambda;\sigma\rangle.\end{aligned} \tag{37}$$

By a slight rearrangement we can also write

$$\begin{aligned}\left(\widetilde{\mu}^{(\lambda)}_\sigma(p,q)\right)^*\widetilde{\psi}^{(\lambda)}_\sigma(p,q) &= \left(\frac{1}{1-\lambda^2}\right)\\ &\quad\times\int \mathrm{d}\tau'(|\psi\rangle\langle\mu|)_{(p',q')}(|\lambda p,-\lambda q,\lambda;\sigma\rangle\langle\lambda p,-\lambda q,\lambda;\sigma|)_{(p'-p,q'-q)}.\end{aligned} \tag{38}$$

Setting $|\mu\rangle = |\psi\rangle$, generalizing from $|\psi\rangle\langle\psi|$ to the density matrix $\hat{\rho} = \sum w_\psi|\psi\rangle\langle\psi|$, and dividing by h gives the 'diagonal' component of this sesquilinear form, the generalized Q-function. Non-negative by construction, from (37) and (38) it is

$$\begin{aligned}\widetilde{Q}^{(\lambda)}_\sigma(p,q;\rho) &\equiv \frac{1}{h}\sum_\psi w_\psi\left|\widetilde{\psi}^{(\lambda)}_\sigma(p,q)\right|^2\\ &= \frac{1}{h}\left(\frac{1}{1-\lambda^2}\right)\langle p,q,\lambda;\sigma|\hat{\rho}|p,q,\lambda;\sigma\rangle\\ &= \left(\frac{1}{1-\lambda^2}\right)\int \mathrm{d}p'\,\mathrm{d}q'\rho(p',q')\rho^{(\lambda)}_\sigma(p;p'-p,q'-q),\end{aligned} \tag{39}$$

where

$$\rho(p,q) = \frac{1}{h}(\hat{\rho})_{(p,q)} = \frac{1}{h}\mathrm{Tr}(\hat{\rho}\hat{\Delta}(p,q)) \tag{40}$$

is the Wigner function for the state $\hat{\rho}$, $|p,q,\lambda;\sigma\rangle$ is given by (36), and $\rho^{(\lambda)}_\sigma$ is the Wigner function corresponding to the p and q dependent squeezed state $|\lambda p,-\lambda q,\lambda;\sigma\rangle$:

$$\rho^{(\lambda)}_\sigma(p;p'-p,q'-q) = \frac{1}{h}(|\lambda p,-\lambda q,\lambda;\sigma\rangle\langle\lambda p,-\lambda q,\lambda;\sigma|)_{(p'-p,q'-q)}. \tag{41}$$

The multiplier $1/h$ is chosen by convention so that $\widetilde{Q}^{(\lambda)}_\sigma(p',q';\rho)$, $\rho(p',q')$ and $\rho^{(\lambda)}_\sigma(p;p'-p,q'-q)$ are all normalized with respect to the integral $\int \mathrm{d}p'\mathrm{d}q'$.

4. Discussion

When there is no squeezing of the fiducial state then $\lambda \to 0$, $|\lambda p,-\lambda q,\lambda;\sigma\rangle \to |\sigma\rangle$, and $|p,q,\lambda;\sigma\rangle \to |p,q;\sigma\rangle$ (defined in equation (2)). In that case

$$\begin{aligned}\widetilde{Q}^{(0)}_\sigma(p,q;\rho) \equiv \widetilde{Q}_\sigma(p,q;\rho) &= \frac{1}{h}\langle p,q;\sigma|\hat{\rho}|p,q;\sigma\rangle\\ &= \int \mathrm{d}p'\,\mathrm{d}q'\rho(p',q')\rho_\sigma(p'-p,q'-q),\end{aligned} \tag{42}$$

where

$$\rho(p,q) = \frac{1}{h}\mathrm{Tr}(\hat{\rho}\Delta(p,q))$$

and

$$\rho_\sigma(p,q) = \frac{1}{h}\,\mathrm{Tr}(|\sigma\rangle\langle\sigma|\Delta(p,q))$$

are the Wigner functions corresponding to the states $\hat{\rho}$ and $|\sigma\rangle\langle\sigma|$.

When $|\sigma\rangle$ is the vacuum state, $\widetilde{Q}_\sigma(p,q;\sigma)$ is the well-known Husimi or Q-function and the content of equations (42) is well known, [9, 11, 20, 22]. Generalized to an arbitrary fiducial state $|\sigma\rangle$ the first of equations (42) says that the Q-function is the expectation of state $\hat{\rho}$ with respect to the coherent state of equation (2) while the second expresses it as the Wigner function for $\hat{\rho}$ smeared with respect to the Wigner function for $|\sigma\rangle\langle\sigma|$. The two Weyl transformed functions which are convoluted in equation (42) each separately corresponds to the scheme of equation (13) with $f(\xi,\eta) = 1$, and $\widetilde{Q}_\sigma(p,q;\sigma)$ itself is also a member of that class of correspondences. This follows the convolution theorem. For instance, we can define the Fourier component, $[\hat{A}]_{(p,q)}$, of the Weyl transform $(\hat{A})_{(p,q)}$ of an operator $\hat{A}$, as

$$[\hat{A}]_{(p,q)} = \int \frac{\mathrm{d}p'\,\mathrm{d}q'}{h}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p'q+q'p)}(\hat{A})_{(p',q')},$$

and using this in equation (42) gives

$$\widetilde{Q}_\sigma(p,q;\sigma) = f^{-1}(-\mathrm{i}\partial_q, -\mathrm{i}\partial_p)\rho(p',q'), \tag{43}$$

where

$$f^{-1}(\xi,\eta) = [|\sigma\rangle\langle\sigma|]_{(\hbar\xi,\hbar\eta)}.$$

The customary choice for the fiducial state is the vacuum [9, 11]. In particular, for a harmonic oscillator in the ground state $|\sigma\rangle = |0\rangle$, where

$$\langle x|0\rangle = \frac{\alpha^{1/2}}{\pi^{1/4}}\,\mathrm{e}^{-\frac{1}{2}\alpha^2x^2}, \qquad \text{and} \qquad \alpha^2 = \frac{m\omega}{\hbar},$$

which gives for the Weyl transform of $|0\rangle\langle 0|$ and its Fourier component

$$(|0\rangle\langle 0|)_{(p,q)} = 2\,\mathrm{e}^{-\alpha^2q^2}\,\mathrm{e}^{-\frac{p^2}{\alpha^2\hbar^2}} \qquad \text{and} \qquad [|0\rangle\langle 0|]_{(\hbar\xi,\hbar\eta)} = \mathrm{e}^{-\frac{\xi^2}{4\alpha^2}}\,\mathrm{e}^{-\frac{\alpha^2\hbar^2\eta^2}{4}}.$$

Thus, even when there is no squeezing (i.e. $\lambda = 0$) what was a Weyl association $f = 1$ (equation (12)) for the phase space wavefunction $|\psi\rangle\langle h_0|$ becomes an association

$$f(\xi,\eta) = \mathrm{e}^{\frac{\xi^2}{4\alpha^2}}\,\mathrm{e}^{\frac{\alpha^2\hbar^2\eta^2}{4}} \tag{44}$$

for the Q-function, equation (43). Although the function $f(\xi,\eta)$ of equation (44) does not have the properties $f(0,\eta) = 1 = f(\xi,0)$ the distribution $\widetilde{Q}_\sigma(p,q;\sigma)$ which it generates is non-negative. It is a positive operator-valued measure (POM) [26]. This association is a special case of the s-family of orderings considered by Cahill and Glauber [21, 22], which in the notation of this paper can be written

$$f^{(s)}(\xi,\eta) = \mathrm{e}^{s\frac{\xi^2}{4\alpha^2}}\,\mathrm{e}^{s\frac{\alpha^2\hbar^2\eta^2}{4}}.$$

For $\lambda \neq 0$ the functions $\widetilde{Q}^{(\lambda)}_\sigma(p,q;\rho)$, equation (39), are also a POMs, but owing to the extra p-dependence of the smoothing function they do not have corresponding functions $f(\xi,\eta)$. The form of equation (39) shows that $\widetilde{Q}^{(\lambda)}_\sigma$ is proportional to the average of $\hat{\rho}$ with respect to the state $|p,q,\lambda;\sigma\rangle$, equation (36). In other words, $\widetilde{Q}^{(\lambda)}_\sigma$ is proportional to the probability of finding the system in the state $|p,q,\lambda;\sigma\rangle$ when it has been prepared in the state $\hat{\rho}$. The state $|p,q,\lambda;\sigma\rangle$ is a minimum uncertainty squeezed state when the fiducial state $|\sigma\rangle$

is the vacuum state. To see this using (25), (33) and (36) it is easy to show that, for any choice of $|\sigma\rangle$,

$$\langle p, q, \lambda; \sigma|\hat{p}|p, q, \lambda; \sigma\rangle = \frac{p}{1-\lambda} + \left(\frac{1+\lambda}{1-\lambda}\right)\langle\sigma|\hat{p}|\sigma\rangle,$$
$$\langle p, q, \lambda; \sigma|\hat{q}|p, q, \lambda; \sigma\rangle = \frac{q}{1+\lambda} + \left(\frac{1-\lambda}{1+\lambda}\right)\langle\sigma|\hat{q}|\sigma\rangle,$$

and that for momentum and position the standard deviations for this state are

$$\Sigma_\sigma^{(\lambda)}(p) = \left|\frac{1+\lambda}{1-\lambda}\right| \Sigma_\sigma(p) \qquad \text{and} \qquad \Sigma_\sigma^{(\lambda)}(q) = \left|\frac{1-\lambda}{1+\lambda}\right| \Sigma_\sigma(q),$$

where $\Sigma_\sigma(p)$ and $\Sigma_\sigma(q)$ are the corresponding standard deviations for state $|\sigma\rangle$. So, whatever the degree of squeezing, $\Sigma_\sigma^{(\lambda)}(p)\Sigma_\sigma^{(\lambda)}(q) = \Sigma_\sigma(p)\Sigma_\sigma(q)$, and for the vacuum state this product is the minimum value $\hbar/2$.

We could equally choose to work with $\psi_\sigma^{(\lambda)}$ instead of $\widetilde{\psi}_\sigma^{(\lambda)}$. For instance

$$Q_\sigma^{(\lambda)}(p, q; \rho) \equiv \frac{1}{h}\sum_\psi w_\psi \left|\psi_\sigma^{(\lambda)}(p, q)\right|^2 \tag{45}$$

is directly related to $\widetilde{Q}_\sigma^{(\lambda)}(p, q; \rho)$, for from equations (16), (17), (20) and (21) we have

$$\widetilde{\psi}_\sigma^{(\lambda)}(p, q) = \tfrac{1}{2}\langle\sigma|\hat{\Pi}\hat{\Delta}^{(\lambda)\dagger}(p/2, q/2)|\psi\rangle = \tfrac{1}{2}\psi_{\sigma_r}^{(-\lambda)}(p/2, q/2), \tag{46}$$

so that

$$\psi_\sigma^{(\lambda)}(p, q) = 2\widetilde{\psi}_{\sigma_r}^{(-\lambda)}(2p, 2q), \tag{47}$$

where $|\sigma_r\rangle = \hat{\Pi}|\sigma\rangle$ is a reflected fiducial state.

The time dependence of $\widetilde{Q}_\sigma^{(\lambda)}(p, q; \rho)$—or of $Q_\sigma^{(\lambda)}(p, q; \rho)$—enters through the time dependence of $\hat{\rho}$, for instance via its Weyl transform $h \times \rho(p', q')$ in the third of equations (39). The equation of motion of Wigner functions is well known [5] and can be transferred to $\widetilde{Q}_\sigma^{(\lambda)}(p, q; \rho)$ itself by partial integration in equation (39). Another way would be to find the time dependence of $\widetilde{\psi}_\sigma^{(\lambda)}(p, q)$ itself which enters through the time dependence of $|\psi\rangle$. In [18] it was chosen to study the time variation of $\psi_\sigma^{(\lambda)}(p, q)$—as it is itself a Weyl transform and, from that standpoint, basic—when driven by a Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{q})$. According to equations (46) and (47), this knowledge transfers to $\widetilde{\psi}_\sigma^{(\lambda)}(p, q)$.

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